

# Positivity of the time constant in a continuous model of first passage percolation

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**Abstract:** We consider a non trivial Boolean model  $\Sigma$  on  $\mathbb{R}^d$  for  $d \geq 2$ . For every  $x, y \in \mathbb{R}^d$  we define  $T(x, y)$  as the minimum time needed to travel from  $x$  to  $y$  by a traveler that walks at speed 1 outside  $\Sigma$  and at infinite speed inside  $\Sigma$ . By a standard application of Kingman sub-additive theorem, one easily shows that  $T(0, x)$  behaves like  $\mu\|x\|$  when  $\|x\|$  goes to infinity, where  $\mu$  is a constant named the time constant in classical first passage percolation. In this paper we investigate the positivity of  $\mu$ . More precisely, under an almost optimal moment assumption on the radii of the balls of the Boolean model, we prove that  $\mu > 0$  if and only if the intensity  $\lambda$  of the Boolean model satisfies  $\lambda < \hat{\lambda}_c$ , where  $\hat{\lambda}_c$  is one of the classical critical parameters defined in continuum percolation.

*Keywords :* Boolean model, continuum percolation, first passage percolation, critical point, time constant.

## 1 Introduction and main results

### 1.1 Boolean model

The Boolean model is defined as follows. At each point of a homogeneous Poisson point process on the Euclidean space  $\mathbb{R}^d$ , we center a ball of random radius. We assume that the radii of the balls are independent, identically distributed and independent of the point process. The Boolean model is the union of the balls. There are three parameters:

- An integer  $d \geq 2$ . This is the dimension of the ambient space  $\mathbb{R}^d$ .
- A real number  $\lambda > 0$ . The intensity measure of the Poisson point process of centers is  $\lambda|\cdot|$  where  $|\cdot|$  denotes the Lebesgue measure on  $\mathbb{R}^d$ .
- A probability measure  $\nu$  on  $(0, +\infty)$ . This is the common distribution of the radii.

We will denote the Boolean model by  $\Sigma(\lambda, \nu, d)$  or  $\Sigma$ .

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More precisely, the Boolean model is defined as follows. Let  $\xi$  be a Poisson point process on  $\mathbb{R}^d \times (0, +\infty)$  with intensity measure  $\lambda|\cdot| \otimes \nu$ . Set

$$\Sigma(\lambda, \nu, d) = \bigcup_{(c,r) \in \xi} B(c, r)$$

where  $B(c, r)$  denotes the open Euclidean ball of  $\mathbb{R}^d$  with center  $c$  and radius  $r$ . We refer to the book by Meester and Roy [13] for background on the Boolean model. We also denote by  $S(c, r)$  the Euclidean sphere of  $\mathbb{R}^d$  with center  $c$  and radius  $r$ . We write  $S(r)$  when  $c = 0$ .

In this paper, we will always assume

$$\int_{(0, +\infty)} r^d \nu(dr) < \infty. \quad (1)$$

When (1) does not hold, all the models we consider are trivial. This is due to the fact that, if (1) does not hold, then for any  $\lambda > 0$ , with probability one,  $\Sigma = \mathbb{R}^d$ . This is Proposition 3.1 in [13].

Let us state a simple consequence of (1). With probability one, the number of random balls which touch a given bounded subset of  $\mathbb{R}^d$  is finite<sup>1</sup>.

Let  $\chi$  denote the set of centers, that is the projection of  $\xi$  on  $\mathbb{R}^d$ . This is a Poisson point process of intensity measure  $\lambda|\cdot|$ . For each  $c \in \chi$ , we denote by  $r(c)$  the unique<sup>2</sup> real  $r$  such that  $(c, r)$  belongs to  $\xi$ . When  $c \in \mathbb{R}^d \setminus \chi$ , we set  $r(c) = 0$ .

## 1.2 Paths

In this paper we only consider polygonal paths. A path is a finite sequence of *distinct* points of  $\mathbb{R}^d$  - if the points are not distinct, we simply name it a sequence. The length of a path  $\pi = (x_0, \dots, x_k)$  is

$$\ell(\pi) = \sum_{i=1}^k \|x_i - x_{i-1}\|$$

where  $\|\cdot\|$  denotes the usual Euclidean norm on  $\mathbb{R}^d$ . In some cases, we will also see  $\pi$  as a curve  $[0, \ell(\pi)] \rightarrow \mathbb{R}^d$  parametrized by arc length. A path from  $A \subset \mathbb{R}^d$  to  $B \subset \mathbb{R}^d$  is a path such that  $\pi(0) \in A$  and  $\pi(\ell(\pi)) \in B$ . A path is in  $C \subset \mathbb{R}^d$  if  $\pi([0, \ell(\pi)]) \subset C$ . Notice that if  $\pi = (x_0, \dots, x_k)$  is a path, then  $\pi([0, \ell(\pi)])$  is the finite union of the closed segments  $[x_{i-1}, x_i]$  for  $i \in \{1, \dots, k\}$ . All these definitions can be extended to sequences in a natural way.

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1. Let  $n \geq 1$ . The number  $N_n$  of random balls which touch  $B(0, n)$  is a Poisson random variable with parameter

$$\lambda \int_{(0, +\infty)} v_d(r+n)^d \mu(dr)$$

where  $v_d$  is the volume of the unit ball of  $\mathbb{R}^d$ . Therefore, with probability one, all the  $N_n$  are finite.

2. Consider the projection from  $\mathbb{R}^d \times (0, +\infty) \rightarrow \mathbb{R}^d$ . With probability one, the restriction to  $\xi$  of this projection is one-to-one.

### 1.3 Percolation in the Boolean model

**Two critical thresholds.** If  $A$  and  $B$  are two subsets of  $\mathbb{R}^d$ , we set

$$\{A \xleftrightarrow{\Sigma} B\} = \{\text{There exists a path in } \Sigma \text{ from } A \text{ to } B\}$$

and

$$\{0 \xleftrightarrow{\Sigma} \infty\} = \{\text{The connected component of } \Sigma \text{ that contains the origin is unbounded}\}.$$

We define two critical thresholds by

$$\lambda_c = \lambda_c(\nu, d) = \sup\{\lambda > 0 : P(0 \xleftrightarrow{\Sigma} \infty) = 0\} \in [0, +\infty]$$

and

$$\widehat{\lambda}_c = \widehat{\lambda}_c(\nu, d) = \sup\{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\} \in [0, +\infty]$$

where  $S(u)$ ,  $u > 0$ , denotes the Euclidean sphere of radius  $u$  centered at the origin.

**Non triviality of the thresholds.** Recall that we assume (1). The thresholds are non trivial. More precisely,

$$0 < \widehat{\lambda}_c \leq \lambda_c < \infty.$$

The inequality  $\lambda_c < \infty$  is proven for a more general model by Hall in [9] (see Theorem 3). In our setting, this can be proven in a simple way by coupling the Boolean percolation model with a Bernoulli percolation model on  $\mathbb{Z}^d$ . This is explained in the remark below the proof of Theorem 3.3 in the book by Meester and Roy [13]. The inequality  $\widehat{\lambda}_c \leq \lambda_c$  is a consequence of the following simple fact:

$$P(0 \xleftrightarrow{\Sigma} \infty) = \lim_{r \rightarrow \infty} P(0 \xleftrightarrow{\Sigma} S(2r)) \leq \limsup_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)).$$

The proof of the inequality  $0 < \widehat{\lambda}_c$  is implicit in [6] where one of the main aims is to prove the positivity of  $\lambda_c$ . We refer to Appendix A for more details.

**The set  $\{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\}$  is open.** This result is implicit in [6]. We refer to Appendix A for more details.

**Phase transition.** In particular,  $\lambda_c$  is non trivial. Therefore, there exists a subcritical phase and a supercritical phase for percolation.

- If  $\lambda < \lambda_c$ , then with probability one there is no unbounded component in  $\Sigma$ .
- If  $\lambda > \lambda_c$ , then with probability one there exists at least one (and actually a unique) unbounded component in  $\Sigma$ .

We refer to [13] for background on percolation in the Boolean model.

**Sharp threshold.** The critical parameter  $\lambda_c$  is probably the more intuitive to define, however in what follows the relevant critical parameter to consider is  $\hat{\lambda}_c$ . This is the reason why, even if we do not use these results in what follows, we gather here known results concerning the link between  $\lambda_c$  and  $\hat{\lambda}_c$ .

If the radii are bounded, then  $\lambda_c = \hat{\lambda}_c$ . This is a sharp threshold property. The sharpness of the transition in the discrete setting was proved independently by Menshikov [14] and by Aizenman-Barsky [2]. The first proof of the equality  $\lambda_c = \hat{\lambda}_c$  relied on the analogous result in the discrete setting. We refer to [13] for the proof (see Theorem 3.5) and references. Ziesche gives in [15] a short proof of the equality  $\lambda_c = \hat{\lambda}_c$  for bounded radii. It relies on a new and short proof of the analogous result in the discrete setting by Duminil-Copin and Tassion [4, 5].

In dimension 2, the sharpness of the transition is one of the results proven recently by Ahlberg, Tassion and Teixeira in [1], using a strategy which is specific to the dimension 2.

## 1.4 First-passage percolation in the Boolean model

In [8], Régine Marchand and the first author studied a model introduced by Deijfen in [3]. The model we introduce in this paper appears implicitly in [8] as an intermediate model. We refer to [8] for the definition of Deijfen's model and its links with the model defined here.

A traveler walks on  $\mathbb{R}^d$ . Inside the Boolean model  $\Sigma$  he walks at infinite speed. Outside the Boolean model  $\Sigma$  he walks at speed 1. He travels from  $x \in \mathbb{R}^d$  to  $y \in \mathbb{R}^d$  as fast as he can. We denote by  $T(x, y)$  the time needed to perform this travel. For example if  $x$  and  $y$  belong to the same connected component of  $\Sigma$ , then  $T(x, y) = 0$ .

Here is a more formal definition. For any  $a$  and  $b$  in  $\mathbb{R}^d$ , we define  $\tau(a, b)$  as the 1-dimensional Lebesgue measure of  $[a, b] \setminus \Sigma$ . With each path  $\pi = (x_0, \dots, x_n)$  is associated a time as follows:

$$\tau(\pi) = \sum_{i=1}^n \tau(x_{i-1}, x_i).$$

If  $x$  and  $y$  are two points of  $\mathbb{R}^d$ , then  $T(x, y)$  is defined by:

$$T(x, y) = \inf\{\tau(\pi) : \pi \in \mathcal{C}(x, y)\},$$

where  $\mathcal{C}(x, y)$  is the set of paths from  $x$  to  $y$ .

A standard application of Kingman sub-additive theorem yields the following result.

**Theorem 1** *There exists a constant  $\mu = \mu(\lambda, \nu, d) \in [0, 1]$  such that:*

$$\lim_{\|x\| \rightarrow \infty} \frac{T(0, x)}{\|x\|} = \mu \text{ with probability 1 and in } L^1.$$

We emphasize the fact that the convergence stated in Theorem 1 is uniform in all directions. We refer to Appendix B for a proof. For any  $A, B \subset \mathbb{R}^d$  we write

$$T(A, B) = \inf_{a \in A, b \in B} T(a, b).$$

For any  $r > 0$ , we use the shorthand notation

$$T(r) = T(\{0\}, S(r)).$$

By Theorem 1 we get

$$\lim_{r \rightarrow \infty} \frac{T(r)}{r} = \mu \text{ a.s. and in } L^1. \quad (2)$$

## 1.5 Link between percolation and first passage percolation; main result

Consider the following condition:

$$\int_{(0, +\infty)} \nu([r, +\infty))^{1/d} dr < \infty. \quad (3)$$

It appears in the paper by Martin [12] about greedy lattice paths and animals. We refer to [12] for a discussion about Condition (3). For example, for any  $\varepsilon > 0$ ,

$$\int_{(0, +\infty)} r^d \ln_+(r)^{d-1+\varepsilon} \nu(dr) < \infty \Rightarrow \int_{(0, +\infty)} \nu([r, +\infty))^{1/d} dr < \infty \Rightarrow \int_{(0, +\infty)} r^d \nu(dr) < \infty.$$

Here is the main result of this paper.

**Theorem 2** *Assume (3). Let  $\lambda > 0$ . Then*

$$\mu(\lambda, \nu, d) = 0 \text{ if and only if } \lambda \geq \widehat{\lambda}_c(\nu, d).$$

Since the set  $\{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\}$  is open (see Appendix A), Theorem 2 is in fact equivalent to the following proposition, that we actually prove in the next sections.

**Proposition 3** *Assume (3). Let  $\lambda > 0$ . Then*

$$\mu(\lambda, \nu, d) > 0 \text{ if and only if } \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0.$$

Let us define a new threshold by

$$\lambda_\mu = \lambda_\mu(\nu, d) = \sup\{\lambda > 0 : \mu(\lambda, \nu, d) = 0\}.$$

Theorem 2 can be reformulated to obtain the following corollary.

**Corollary 4** *Assume (3). Then*

$$\lambda_\mu = \widehat{\lambda}_c.$$

Moreover,

$$\mu(\lambda_\mu(\nu, d), \nu, d) = 0.$$

Theorem 2 is analogous to the result of Kesten [10] (Theorem 6.1) in the framework of Bernoulli percolation and first passage percolation on  $\mathbb{Z}^d$ . The proof of Kesten can be adapted in our setting in the case of bounded radii. In the general case, some further arguments are needed.

In [8], the following result was implicitly proved: if (3) holds, then  $\mu(\lambda, \nu, d)$  is positive for small enough  $\lambda > 0$ . Theorem 2 is therefore a strengthening of this result.

## 2 Proof of $\mu > 0 \Rightarrow \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0$

Let  $\lambda > 0$ . Let us first prove that

$$\lim_{r \rightarrow \infty} \frac{T(S(r), S(2r))}{r} = \mu \text{ a.s.} \quad (4)$$

Any path from 0 to  $S(2r)$  can be seen as the concatenation of a first path from 0 to  $S(r)$  and a second path from  $S(r)$  to  $S(2r)$ . Taking infimums, we get

$$T(0, S(r)) + T(S(r), S(2r)) \leq T(0, S(2r)). \quad (5)$$

On the other hand, for any  $x$  in  $S(r)$  we have

$$\begin{aligned} T(0, S(2r)) &\leq T(0, x) + T(x, S(2r)) \\ &\leq \left( \sup_{x' \in S(r)} T(0, x') \right) + T(x, S(2r)). \end{aligned}$$

Taking the infimum in  $x$ , we now get

$$T(0, S(2r)) \leq \left( \sup_{x' \in S(r)} T(0, x') \right) + T(S(r), S(2r)). \quad (6)$$

From (5) and (6) we get

$$T(0, S(2r)) - \left( \sup_{x' \in S(r)} T(0, x') \right) \leq T(S(r), S(2r)) \leq T(0, S(2r)) - T(0, S(r)).$$

By Theorem 1 we then deduce (4).

Now assume  $\mu(\lambda, \nu, d) > 0$ . Let us prove  $\lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0$ . For all  $r > 0$ ,

$$P(S(r) \xleftrightarrow{\Sigma} S(2r)) \leq P\left(\frac{T(S(r), S(2r))}{r} = 0\right).$$

But this tends to 0 as  $r$  tends to infinity thanks to (4) and the assumption  $\mu > 0$ .

## 3 Proof of $\lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0 \Rightarrow \mu > 0$

Let  $\lambda > 0$  such that  $\lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0$ . In this section, we prove that this implies  $\mu(\lambda, \nu, d) > 0$ . Let us give the plan of the proof. First fix a large enough  $A$ . Since  $\lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0$ , we can suppose that with high probability the time needed to cross an annulus of the form  $B(x, 2A) \setminus B(x, A)$  is bigger than some positive constant. We consider now a very large  $r$ , a lot bigger than  $A$ . We consider a nice geodesic  $\pi$  from 0 to the Euclidean sphere  $S(r)$ . Say, for concreteness, that a random ball of the Boolean model is large if its radius is larger than  $10A$ .

First, imagine that there exists no large random balls. The idea is then to discretize the geodesic  $\pi$  in a convenient way at scale  $A$ : the traveler, when moving along the geodesic, crosses several annuli of the form  $B(x, 2A) \setminus B(x, A)$ . As explained previously, each time the traveler crosses such an annulus, this costs him some amount of time with high probability. As there is no large balls, what occurs in far enough annuli is independent and the positivity of  $\mu$  follows.

The difficulty is to take care of large balls. If  $\tau_A(\pi)$  denotes the time needed to travel along the path  $\pi$  when we throw away large balls, then we can write

$$T(r) = \tau(\pi) = \tau_A(\pi) - (\tau_A(\pi) - \tau(\pi)).$$

We then have to give for the perturbation  $(\tau_A(\pi) - \tau(\pi))$  an upper bound which is smaller than the lower bound we obtain for  $\tau_A(\pi)$ . This is achieved by relating the perturbation to the greedy paths model and by working on the discretization of our path.

### 3.1 Greedy paths

Let  $\pi = (x_0, \dots, x_k)$  be a path. Recall that a path is a family of distinct points of  $\mathbb{R}^d$ . Set

$$r(\pi) = \sum_{1 \leq i \leq k} r(x_i)$$

(recall that  $r(x)$  is defined at the end of the paragraph about the Boolean model) and

$$S = \sup_{\pi} \frac{r(\pi)}{\ell(\pi)}$$

where the supremum runs over all paths such that  $x_0 = 0$  and  $k \geq 1$ . Note that the intensity measure of the underlying Poisson point process  $\xi$  is  $\lambda|\cdot| \otimes \nu = |\cdot| \otimes \lambda\nu$ .

We can consider the greedy path model when the intensity measure is  $|\cdot| \otimes m$  where  $m$  is a given finite measure on  $(0, +\infty)$ . In that case, we write  $r_m(\pi)$  and  $S_m$ .

**Theorem 5 ([8])** *Let  $m$  be a finite measure on  $(0, +\infty)$ . There exists a constant  $C = C(d)$  such that*

$$E(S_m) \leq C \int_{(0, \infty)} m((r, +\infty))^{1/d} dr.$$

This is a consequence of (11) in [8] and Lemma 2.1 in the same article. Note that the results requires the assumption  $d \geq 2$ . The result is the analogue in the continuous setting of a result by Martin [12] in the discrete setting.

### 3.2 Constants

Fix  $\eta = \eta(d) > 0$  such that

$$\mathbb{R}^d \subset \bigcup_{i \in \mathbb{Z}^d} B(\eta i, 1). \quad (7)$$

Let  $K = K(d)$  be the cardinality of  $B(0, 12\eta^{-1}) \cap \mathbb{Z}^d$ . Fix  $\kappa = \kappa(d) > 0$  large enough and  $\varepsilon = \varepsilon(d) > 0$  small enough such that

$$K \exp(-\kappa/2) + K \exp(\kappa/2) \varepsilon \leq 1/2. \quad (8)$$

This will be used as follows. Let  $k \geq 1$ . Let  $\mathcal{S}(k)$  be the set of sequences  $(s(0), \dots, s(k-1))$  of distinct elements of  $\mathbb{Z}^d$  such that  $s(0) = 0$  and, for any  $i \in \{1, \dots, k-1\}$ ,  $\|s(i) - s(i-1)\| \leq 12\eta^{-1}$ . Let  $(Z_x)_{x \in \mathbb{Z}^d}$  be a family of random variables with Bernoulli distribution of parameter at least  $1 - \varepsilon$ . Assume that for any  $(s(0), \dots, s(k-1))$  in  $\mathcal{S}(k)$ , the family  $(Z_{s(0)}, \dots, Z_{s(k-1)})$  is independent. Then

$$P \left( \inf_{(s(0), \dots, s(k-1)) \in \mathcal{S}(k)} \frac{1}{k} \sum_{i=0}^{k-1} Z_{s(i)} \geq \frac{1}{2} \right) \geq 1 - \frac{1}{2^k}. \quad (9)$$

Let us prove this inequality. If the infimum is smaller than  $1/2$ , then there exists at least one path  $s \in \mathcal{S}(k)$  such that

$$\frac{1}{k} \sum_{i=0}^{k-1} Z_{s(i)} \leq \frac{1}{2}.$$

But the probability of such an event is bounded from above by

$$P \left( \frac{1}{k} \sum_{i=0}^{k-1} Z'_i \leq \frac{1}{2} \right)$$

where the  $Z'_i$  are i.i.d.r.v. with Bernoulli distribution of parameter  $1 - \varepsilon$ . Therefore

$$\begin{aligned} P \left( \inf_{(s(0), \dots, s(k-1)) \in \mathcal{S}(k)} \frac{1}{k} \sum_{i=0}^{k-1} Z_{s(i)} \leq \frac{1}{2} \right) &\leq \text{card}(\mathcal{S}(k)) P \left( \sum_{i=0}^{k-1} Z'_i \leq k/2 \right) \\ &\leq K^{k-1} \left( \exp(\kappa/2) ((1 - \varepsilon) \exp(-\kappa) + \varepsilon) \right)^k \\ &\leq (K \exp(-\kappa/2) + K \exp(\kappa/2) \varepsilon)^k \\ &\leq \frac{1}{2^k} \end{aligned}$$

by (8).

Fix  $A = A(\lambda, \nu, d) > 0$  large enough such that

$$P(S(A) \xleftrightarrow{\Sigma} S(2A)) \leq \varepsilon/2$$

and

$$C \int_{(0, +\infty)} \lambda^{1/d} \nu_A((r, +\infty))^{1/d} dr \leq 20000^{-1} \quad (10)$$

where  $C$  is the constant which appears in Theorem 5 and where the measure  $\nu_A$  is defined by  $\nu_A(\cdot) = \nu(\cdot \cap A]$ . Since<sup>3</sup>

$$\{S(A) \xleftrightarrow{\Sigma} S(2A)\}^c \subset \{T(S(A), S(2A)) > 0\}.$$

we have

$$P(T(S(A), S(2A)) > 0) \geq 1 - \varepsilon/2.$$

Therefore we can fix  $\delta = \delta(\lambda, \nu, d) > 0$  such that

$$P(T(S(A), S(2A)) \geq \delta) \geq 1 - \varepsilon. \quad (11)$$

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3. Here is one way to prove the inclusion. There exists a geodesic  $\pi$  such that  $T(S(A), S(2A)) = \tau(\pi)$ . This can be proven for example by adapting and simplifying the proof of Lemma 6. The inclusion  $\{S(A) \xleftrightarrow{\Sigma} S(2A)\}^c \subset \{T(S(A), S(2A)) > 0\}$  follows from the existence of the geodesic.



### 3.3 Geodesics

The aim of the section is to prove the existence of a geodesic from 0 to  $S(r)$  ( $r > 0$ ) that have good properties, as listed in the following lemma.

**Lemma 6** *For any  $r > 0$  there exists a path  $\pi = (x_0, \dots, x_k)$  such that*

- (i).  $x_0 = 0$ ,  $\|x_k\| = r$  and, for all  $i$ ,  $\|x_i\| \leq r$ .
- (ii).  $\tau(\pi) = T(r)$ .
- (iii).  $\ell(\pi)$  is minimal among all paths satisfying (i) and (ii).
- (iv). For any ball  $B = B(c, r(c))$  of the Boolean model,  $\pi^{-1}(B)$  (here we see  $\pi$  as a curve parametrized by arc-length) is an interval of length at most  $2r(c)$  (which can be empty).

We do not need the property (iii) of the geodesic  $\pi$  but we use it as a tool to prove the property (iv). To prove Lemma 6 we use two intermediate lemmas. Let us first prove the following result. Recall that a path is a sequence whose points are distinct.

**Lemma 7** *For any  $r > 0$  and any sequence  $\pi^1$  from 0 to  $S(r)$  there exists a path  $\pi^2 = (x_0, \dots, x_n)$  from 0 to  $S(r)$  such that*

- (a).  $\tau(\pi^2) \leq \tau(\pi^1)$  and  $\ell(\pi^2) \leq \ell(\pi^1)$ .
- (b). For all  $i \in \{0, n\}$ ,  $x_i \in \overline{B(0, r)}$ .
- (c). For all  $i \in \{1, \dots, n-1\}$ , there exists  $(c, \rho) \in \xi$  such that  $x_i \in \overline{B(c, \rho)}$ .
- (d). For all  $(c, \rho) \in \xi$ , there exists at most two indices  $i \in \{1, \dots, n-1\}$  such that  $x_i \in \overline{B(c, \rho)}$ .
- (e).  $n \leq 2N + 2$  where  $N$  denotes the number of random balls of the Boolean model which touch  $\overline{B(0, r)}$ .

**Proof of Lemma 7.** The strategy is simple. We start from the sequence  $\pi^1$ . We then perform a finite number of steps, that are illustrated in Figures 1 and 2. None of them increases the length nor the travel time of the sequence. Moreover, at the end of the procedure, we get a path which fulfills the required properties. For ease of notation, at each step, we write the sequence  $(x_0, \dots, x_n)$  even if the sequence is modified. A point of the sequence is one of the  $x_i$ .

Step 1. If there exists  $i, j \in \{0, n\}$  such that  $x_i = x_j$  and  $i < j$  then we remove points  $x_k$  for  $k \in \{i+1, \dots, j\}$ . We repeat this action until there exists no more such  $i, j$ .

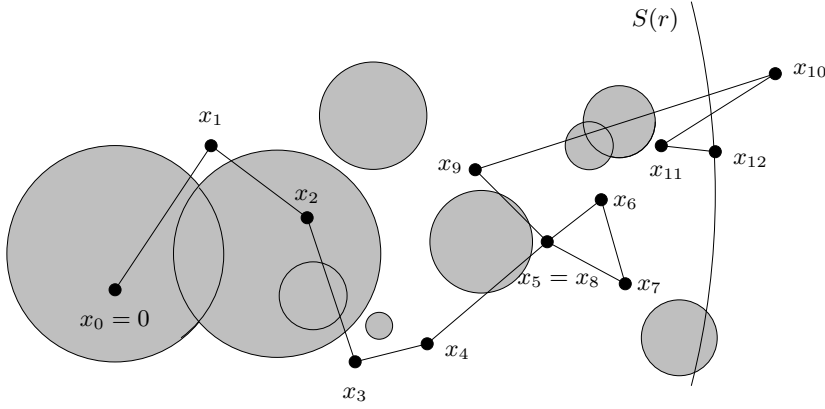
We get a path from 0 to  $S(r)$  which satisfies (a): its length is at most the length of the initial sequence and its travel time is at most the travel time of the initial sequence.

Step 2. Then, we stop the path (seen as a curve) at its first intersection with  $S(r)$ .

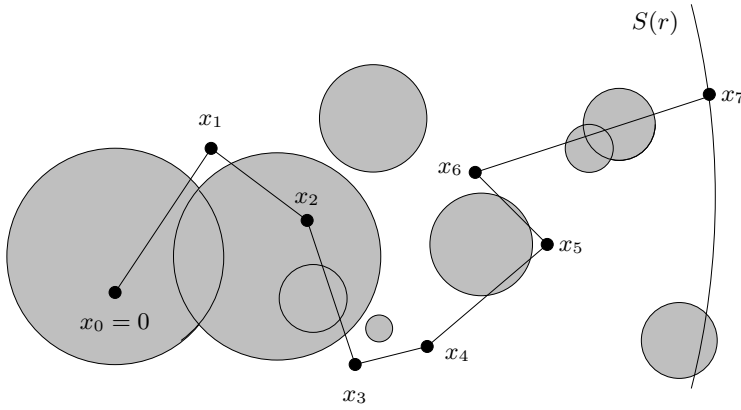
We get a path from 0 to  $S(r)$  satisfying (a) and (b). From now on, we will never change the first nor the last point of the path.

Step 3. Write

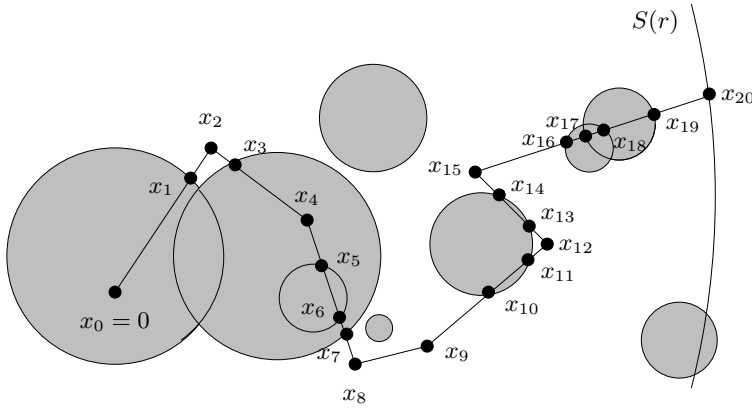
$$\partial = \bigcup_{(c, \rho) \in \xi} S(c, \rho).$$



The sequence  
 $\pi^1 = (x_0, \dots, x_{12})$   
 and the balls of the  
 Boolean model  
 (in grey).



The path  
 $(x_0, \dots, x_7)$   
 obtained after  
 Steps 1 and 2.



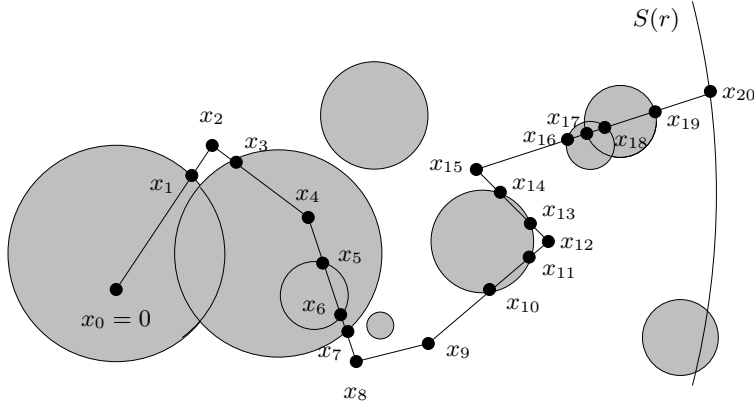
The path  
 $(x_0, \dots, x_{20})$   
 obtained after  
 Step 3.

Figure 1: Evolution of a sequence  $\pi^1$  through Steps 1, 2 and 3 of the procedure.

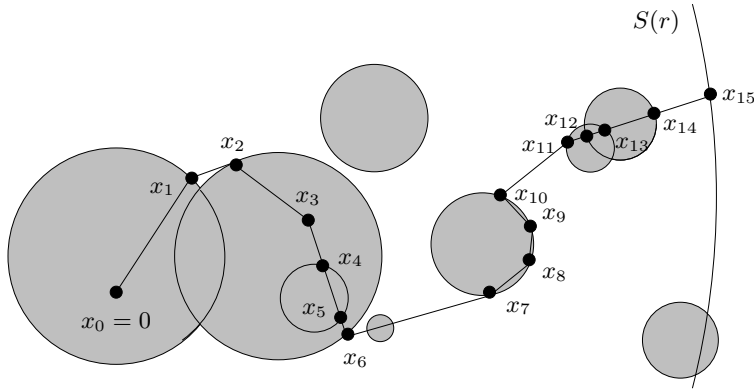
We add successively (when the point is not yet a point of the path) to the path each intersection between the path (seen as a curve) and  $\partial$ . Recall that we work on the full event "the number of random balls of the Boolean model which touch  $\overline{B(0, r)}$  is finite". The number of intersection between the path and  $\partial$  is therefore finite. The new path satisfies (a) and (b).

Step 4. Let  $i, j \in \{1, \dots, n-1\}$  be such that

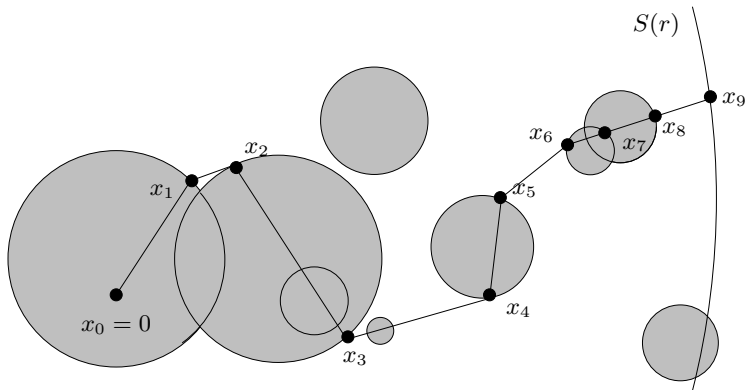
- $i \leq j$  and for all  $k \in \{i, \dots, j\}$ ,  $x_k$  does not belong to  $\overline{\Sigma}$ .



The path  
 $(x_0, \dots, x_{20})$   
 obtained after  
 Step 3.



The path  
 $(x_0, \dots, x_{15})$   
 obtained after  
 Step 4.



The path  
 $(x_0, \dots, x_9)$   
 obtained after  
 Step 5.

Figure 2: Evolution of a sequence  $\pi^1$  through Steps 4 and 5 of the procedure.

- $i - 1 = 0$  or  $x_{i-1} \in \overline{\Sigma}$ .
- $j + 1 = n$  or  $x_{j+1} \in \overline{\Sigma}$ .

Then the path strictly between  $x_{i-1}$  and  $x_{j+1}$  does not touch  $\Sigma$ . Otherwise the curve would touch the boundary  $\partial$ . Therefore, the travel length of the path between  $x_{i-1}$  and  $x_{j+1}$  is at least

$$\|x_i - x_{i-1}\| + \cdots + \|x_{j+1} - x_j\| \geq \|x_{j+1} - x_{i-1}\| \geq \tau(x_{i-1}, x_{j+1}).$$

As a consequence, we can remove points  $x_k$ , for  $k \in \{i, \dots, j\}$  for any such  $i, j$  without increasing the length nor the travel time of the path. We get a path satisfying (a), (b) and (c).

Step 5. We consider successively each point  $x_i$ ,  $i \in \{0, \dots, n\}$  of the path. If there exists  $j \in \{0, n\}$  such that  $x_i$  and  $x_j$  belong to the closure of the same random ball of the Boolean model and if  $j \geq i + 2$ , then  $(x_i, x_j) \subset \Sigma$  and therefore  $\tau(x_i, x_j) = 0$ . As a consequence, we can remove any point  $x_k$  with  $k \in \{i + 1, \dots, j - 1\}$  without increasing the length nor the travel time of the path. We repeat this procedure until there exists no more such  $i$  and  $j$ . In particular, we get a path satisfying (a), (b), (c) and (d). (e) is a consequence of (b), (c) and (d).  $\square$

Let  $N$  be the number of random balls of the Boolean model which touch  $\overline{B(0, r)}$ . We say that a sequence  $(x_0, \dots, x_n)$  is a good sequence if  $n \leq 2N + 2$ ,  $x_0 = 0$ ,  $x_n \in S(r)$  and all the  $x_i$  belongs to  $\overline{B(0, r)}$ . We say that a good sequence  $(x_0, \dots, x_n)$  is a perfect sequence if  $n = 2N + 2$ . Set

$$\mathcal{P}_r = \{\text{perfect sequences}\} = \{0\} \times \overline{B(0, r)}^{2N} \times S(r).$$

We will use Lemma 7 through the following simple consequence.

**Lemma 8** *Let  $r > 0$ .*

- *For any sequence  $\pi^s$  from 0 to  $S(r)$  there exists a good path  $\pi^p$  from 0 to  $S(r)$  such that  $\tau(\pi^p) \leq \tau(\pi^s)$  and  $\ell(\pi^p) \leq \ell(\pi^s)$ .*
- *For any path  $\pi^p$  from 0 to  $S(r)$  there exists a perfect sequence  $\pi^s$  from 0 to  $S(r)$  such that  $\tau(\pi^p) \leq \tau(\pi^s)$  and  $\ell(\pi^p) \leq \ell(\pi^s)$ .*

**Proof of Lemma 8.** The first part is a weak form a Lemma 7. Let us prove the second part. Let  $\pi^p$  be a path from 0 to  $S(r)$ . By Lemma 7 there exists a good path  $(x_0, \dots, x_n)$  from 0 to  $S(r)$  such that  $\tau((x_0, \dots, x_n)) \leq \tau(\pi^p)$  and  $\ell((x_0, \dots, x_n)) \leq \ell(\pi^p)$ . We build a perfect sequence  $\pi^s$  by adding vertices  $x_{n+1} = \dots = x_{2N+2} = x_n$ . The perfect sequence  $\pi^s$  fulfills the required properties.  $\square$

**Proof of Lemma 6.** Here is the idea: (i), (ii) and (iii) follows by some compactness and continuity argument ; (iv) is then a consequence of (iii). Here is a detailed proof.

By Lemma 8,

$$T(r) = \inf_{\pi \in \mathcal{P}_r} \tau(\pi). \quad (12)$$

The set  $\mathcal{P}_r$ , endowed with its usual topology, is compact. The function  $\ell$  is continuous. Let us prove the continuity of  $\tau$  on the full probability event "the number of random balls

that touch any bounded region is finite". It is sufficient to prove the continuity of the map defined by  $(a, b) \rightarrow \tau(a, b)$  where  $a, b \in \mathbb{R}^d$ . For any  $a, b \in \mathbb{R}^d$ ,

$$\tau(a, b) = \|b - a\| \int_0^1 1_{a+t(b-a) \notin \Sigma} dt.$$

Let  $(a, b) \in \mathbb{R}^d$ . If  $a = b$  then the continuity at  $(a, b)$  is straightforward as the time needed to travel a segment is at most the length of the segment. Let us assume  $a \neq b$ . Let  $(a_n)_n$  and  $(b_n)_n$  be two sequences which converge to  $a$  and  $b$ . Write

$$\tau(a_n, b_n) = \|b_n - a_n\| \int_0^1 1_{a_n+t(b_n-a_n) \notin \Sigma} dt.$$

The intersection of  $[a, b]$  with the boundary of  $\Sigma$  is finite. Therefore for almost any  $t \in [0, 1]$ ,  $a + t(b - a)$  does not belong to the boundary of  $\Sigma$  (here we use  $a \neq b$ ). As a consequence, for almost any  $t \in [0, 1]$ ,

$$\lim_{n \rightarrow \infty} 1_{a_n+t(b_n-a_n) \notin \Sigma} = 1_{a+t(b-a) \notin \Sigma}.$$

The result follows by dominated convergence.

By compacity of  $\mathcal{P}_r$  and by continuity of  $\tau$ , there exists a non empty set  $\widehat{\mathcal{P}}_r \subset \mathcal{P}_r$  such that for all  $\pi \in \widehat{\mathcal{P}}_r$ ,  $\tau(\pi) = T(r)$ . Moreover  $\widehat{\mathcal{P}}_r$  is also compact, thus by continuity of  $\ell$  we get the existence of a perfect sequence  $\pi^s$  satisfying (i), (ii) and (iii').  $\ell(\pi^s)$  is minimal among all perfect sequences satisfying (i) and (ii).

By Lemma 7 we get a good path  $\pi^p$  from 0 to  $S(r)$  such that  $\tau(\pi^p) \leq \tau(\pi^s)$  and

$$\ell(\pi^p) \leq \ell(\pi^s). \quad (13)$$

Thanks to (12) and  $\tau(\pi^s) = T(r)$ , the former inequality yields

$$\tau(\pi^p) = T(r).$$

In other words,  $\pi^p$  satisfies (i) and (ii). Let  $\widetilde{\pi}^p$  be a path satisfying (i) and (ii). By Lemma 8 we get a perfect sequence  $\widetilde{\pi}^s$  from 0 to  $S(r)$  such that  $\tau(\widetilde{\pi}^s) \leq \tau(\widetilde{\pi}^p)$  and

$$\ell(\widetilde{\pi}^s) \leq \ell(\widetilde{\pi}^p). \quad (14)$$

Since  $\tau(\widetilde{\pi}^p) = T(r)$ , the former inequality yields

$$\tau(\widetilde{\pi}^s) = T(r).$$

In other words,  $\widetilde{\pi}^s$  is a perfect sequence satisfying (i) and (ii). From (13), (iii') and (14) we get

$$\ell(\pi^p) \leq \ell(\pi^s) \leq \ell(\widetilde{\pi}^s) \leq \ell(\widetilde{\pi}^p)$$

and then

$$\ell(\pi^p) \leq \ell(\widetilde{\pi}^p).$$

Therefore  $\pi^p$  satisfies (iii).

For ease of notation, we now write  $\pi$  for  $\pi^p$ . It remains to check that, as a consequence of (i), (ii) and (iii),  $\pi$  also fulfills (iv). Let us see our path as a curve  $\pi : [0, \ell(\pi)] \rightarrow \mathbb{R}^d$  parametrized by arc-length. Let  $B$  be a random ball that touches the path. Set

$$a = \inf \pi^{-1}(B) \text{ and } b = \sup \pi^{-1}(B).$$

Let  $\tilde{\pi}$  be the path (seen as a curve parametrized by arc-length) obtained by concatenation (with a slight abuse of notation) of  $\pi|_{[0,a]}$ ,  $[\pi(a), \pi(b)]$  and  $\pi|_{[b, \ell(\pi)]}$ . As  $(\pi(a), \pi(b))$  is contained in  $B$  we have  $\tau(\pi(a), \pi(b)) = 0$ . Therefore  $\tau(\tilde{\pi}) \leq \tau(\pi)$  and then  $\tau(\tilde{\pi}) = \tau(\pi)$  by (ii). By (iii) we then get  $\ell(\tilde{\pi}) \geq \ell(\pi)$ . Subtracting the length of the commons parts of the paths, we deduce that the length of  $\pi$  between  $\pi(a)$  and  $\pi(b)$  is at most  $\|\pi(b) - \pi(a)\|$ . As  $\pi$  is parametrized by arc-length, this implies that the paths goes straight from  $\pi(a)$  to  $\pi(b)$ . This implies (iv).  $\square$

### 3.4 Good sites

Set

$$\Sigma_- = \bigcup_{(c,r) \in \xi: r < A} B(c, r).$$

In other words, we throw away all balls of radius larger than  $A$ . Let  $T_-$  be defined from  $\Sigma_-$  in the same way as  $T$  is defined from  $\Sigma$ . In particular  $T_- \geq T$ .

For any  $i \in \mathbb{Z}^d$ , we say that Site  $i$  is good if  $T_-(S(\eta i A, A), S(\eta i A, 2A)) \geq \delta$ . By stationarity, by the inequality  $T_- \geq T$  and by (11), a given site is good with probability at least  $1 - \varepsilon$ . Moreover, as we only consider random balls with radius smaller than  $A$ , the state of Site  $i$  only depends on balls whose center belongs to  $B(\eta i A, 3A)$ . Therefore, the state of Sites  $i$  and  $j$  are independent as soon as  $\|i - j\| \geq 6\eta^{-1}$ .

### 3.5 Skeletons of the geodesic.

Let  $r \geq 20A$ . Let  $\pi$  be the geodesic given by Lemma 6. We see  $\pi$  as a curve  $[0, \ell(\pi)] \rightarrow \mathbb{R}^d$  parametrized by arc length.

**Lemma 9** *Set*

$$k = \lceil r/(20A) \rceil. \tag{15}$$

*There exists a sequence  $0 = t(0) \leq t(1) \leq \dots \leq t(k) \leq \ell(\pi)$  such that*

1. *For any distinct  $j, j' \in \{0, \dots, k\}$ ,  $\|\pi(t(j)) - \pi(t(j'))\| \geq 10A$ .*
2. *For any  $j \in \{1, \dots, k\}$ ,  $\|\pi(t(j)) - \pi(t(j-1))\| \leq 10A$ .*

**Proof.** We first build a sequence  $0 = t_1(0) \leq t_1(1) \leq \dots \leq t_1(k_1) \leq \ell(\pi)$  as follows. Let  $t_1(0) = 0$  and  $k_1 = 0$ . We proceed by induction as follows. At each step we consider the set

$$\left\{ t \in [t_1(k_1), \ell(\pi)] : \pi(t) \notin \bigcup_{j \leq k_1} B(\pi(t_1(j)), 10A) \right\}.$$

- If the set is empty, then the construction is over.

- Otherwise, we define  $t_1(k_1 + 1)$  as the minimum of this non empty set, we increase by one  $k_1$  and the construction goes on.

By construction, the sequence built above fulfills the following properties.

1. For any distinct  $j, j' \in \{0, \dots, k_1\}$ ,  $\|\pi(t_1(j)) - \pi(t_1(j'))\| \geq 10A$ .
2. For any  $j \in \{1, \dots, k_1\}$ , there exists  $j' \in \{0, \dots, j-1\}$  such that  $\|\pi(t_1(j)) - \pi(t_1(j'))\| \leq 10A$ .

Thanks to the stopping criterion, there exists  $k \in \{0, \dots, k_1\}$  such that  $\|\pi(t_k)\| + 10A \geq r$ . Throwing away some points if needed, we can therefore assume that the sequence also satisfies

$$\|\pi(t_1(k_1))\| + 10A \geq r.$$

We now remove the loops in the sequence to get a new sequence  $0 = t_2(0) \leq t_2(1) \leq \dots \leq t_2(k_2) \leq \ell(\pi)$  such that

1. For any distinct  $j, j' \in \{0, \dots, k_2\}$ ,  $|\pi(t_2(j)) - \pi(t_2(j'))| \geq 10A$ .
2. For any  $j \in \{1, \dots, k_2\}$ ,  $\|\pi(t_2(j)) - \pi(t_2(j-1))\| \leq 10A$ .
3.  $t_2(k_2) = t_1(k_1)$ .

We can for example use the following backward construction. For each  $j \in \{1, \dots, k_1\}$  we define  $a(j)$  as the smallest  $j' \in \{0, \dots, j-1\}$  such that  $\|\pi(t_1(j)) - \pi(t_1(j'))\| \leq 10A$ . Let  $k_2$  be the smallest integer  $k$  such that  $a^{(k)}(k_1) = 0$ , where  $a^{(k)}$  is the  $k$  fold composition of  $a$  with itself. Then our new sequence is defined by

$$\begin{aligned} t_2(0) &= t_1(a^{(k_2)}(k_1)) = 0, \\ t_2(1) &= t_1(a^{(k_2-1)}(k_1)), \\ t_2(2) &= t_1(a^{(k_2-2)}(k_1)), \\ &\vdots \\ t_2(k_2) &= t_1(a^{(0)}(k_1)) = t_1(k_1). \end{aligned}$$

It fulfills the required properties.

As  $t_2(k_2) = t_1(k_1)$  we get

$$\|\pi(t_2(k_2))\| + 10A = \|\pi(t_1(k_1))\| + 10A \geq r.$$

Thanks to the second property of the second sequence and to the fact that  $\pi(t_2(0)) = \pi(0) = 0$ , we get

$$10k_2A + 10A \geq r.$$

As  $r \geq 20A$  we get  $k_2 \geq 1$  and then  $20k_2A \geq r$ . Therefore  $k_2 \geq k$ , where  $k = \lceil r/(20A) \rceil$  as defined in the Lemma. Thus, by throwing away, if needed, the last elements of our second sequence, we get a third sequence which fulfills the properties stated in the Lemma.  $\square$

We define a new sequence  $s(0), \dots, s(k)$  which is, in some sense, a discretization of the sequence  $t(0), \dots, t(k)$ . We set  $s(0) = 0$ . For all  $j \in \{1, \dots, k\}$ , we chose  $s(j) \in \mathbb{Z}^d$  such that  $\pi(t(j))$  belongs to  $B(\eta s(j)A, A)$ . This is possible thanks to (7).

**Lemma 10** *The sequence  $s(0), \dots, s(k)$  fulfills the following properties.*

1. *For all  $j \in \{0, \dots, k\}$ ,  $\pi(t(j))$  belongs to  $B(\eta s(j)A, A)$ .*
2. *For all distinct  $j, j' \in \{0, \dots, k\}$ ,  $\|s(j) - s(j')\| \geq 8\eta^{-1}$ .*
3. *For all  $j \in \{1, \dots, k\}$ ,  $\|s(j) - s(j-1)\| \leq 12\eta^{-1}$ .*

**Proof.** The first property holds by definition for any index  $j \geq 1$ . It also holds for  $j = 0$  as  $\pi(t(0)) = \pi(0) = 0$ . Let  $j, j'$  be two distinct elements of  $\{0, \dots, k\}$ . By Lemma 9, we have

$$\|\pi(t(j)) - \pi(t(j'))\| \geq 10A.$$

By definition of  $s(j)$  and  $s(j')$ , we get  $\|\eta s(j)A - \eta s(j')A\| \geq 8A$  and then  $|s(j) - s(j')| \geq 8\eta^{-1}$ . The third property stated in the lemma is proven in the same way.  $\square$

### 3.6 About disturbant balls

For all  $j \in \{0, \dots, k\}$ , we set

$$\Pi(j) = \pi([0, \ell(\pi)]) \cap B(\eta s(j)A, 2A)$$

where  $\pi$  is still the geodesic given by Lemma 6 seen as a curve parametrized by arc length. Let us say that a random ball  $B(c, r(c))$  of the Boolean model disturbs the site  $s(j)$  if  $\Pi(j) \cap B(c, r(c)) \neq \emptyset$  and if  $r(c) \geq A$ . If no random ball disturbs the site  $s(j)$ , then the travel time of the geodesic inside  $B(\eta s(j)A, 2A)$  does not depend on large balls. Set

$$D(j) = \{\text{centers of random balls which disturb site } s(j)\}$$

and

$$D = \bigcup_{j \in \{0, \dots, k\}} D(j).$$

We will use the following upper bound on the number of sites  $s(j)$  which are disturbed by a given large ball of the Boolean model.

**Lemma 11** *Let  $B(c, r(c))$  be a random ball of the Boolean model such that  $r(c) \geq A$ . Let  $j \in \{0, \dots, k\}$ . Then*

$$\text{card}(\{j \in \{0, \dots, k\} : \Pi(j) \cap B(c, r(c)) \neq \emptyset\}) \leq \frac{2r(c)}{A}.$$

**Proof.** The idea is that the geodesic restricted to  $B(c, r(c))$  is an open line segment of length at most  $2r(c)$  and that points in different  $B(\eta s(j)A, 2A)$  are at least at distance  $4A$  from each other. Let us give a detailed proof. For all  $j \in \{0, \dots, k\}$ , we set

$$T(j) = \pi^{-1}(B(\eta s(j)A, 2A)).$$

Note that  $\Pi(j) = \pi(T(j))$ . Set

$$J = \{j \in \{0, \dots, k\} : \pi(T(j)) \cap B(c, r(c)) \neq \emptyset\}.$$



We aim at proving  $\text{card}(J) \leq 2r(c)/A$ .

For all  $j \in J$ , fix  $t'(j) \in T(j)$  such that  $\pi(t'(j)) \in B(c, r(c))$ . As  $t'(j) \in T(j)$ , we have  $\pi(t'(j)) \in B(\eta s(j)A, 2A)$ . For any distinct  $j, j' \in J$ , by Lemma 10, we thus have

$$|\pi(t'(j)) - \pi(t'(j'))| \geq |\eta s(j)A - \eta s(j')A| - 4A \geq 8A - 4A = 4A.$$

As  $\pi$  is parametrized by arc length, we then get

$$|t'(j) - t'(j')| \geq 4A. \quad (16)$$

By Lemma 6,  $\pi^{-1}(B(c, r(c)))$  is an interval of length at most  $2r(c)$ . Ordering the  $t'(j), j \in J$ , using the fact that each such  $t'(j)$  belongs to  $\pi^{-1}(B(c, r(c)))$  and using (16) we thus get

$$2r(c) \geq \max_j t'(j) - \min_j t'(j) \geq 4A(\text{card}(J) - 1).$$

Therefore

$$\text{card}(J) \leq \frac{r(c)}{2A} + 1 \leq \frac{2r(c)}{A}$$

as  $r(c) \geq A$ . □

### 3.7 A second path

We define a new path  $\tilde{\pi}$ . The definition of  $\tilde{\pi}$  is in a sense artificial, since it is built to enable the use of the results on greedy paths.

- It starts from  $\eta s(0)A = 0$  and visits each point of  $D(0)$  (see the subsection about disturbing balls), if any.
- Then it goes to  $\eta s(1)A$  and visits each point of  $D(1)$  it has not visited yet, if any.
- Then it goes to  $\eta s(2)A$  and visits each point of  $D(2)$  it has not visited yet, if any.
- ...
- Then it goes to  $\eta s(k)A$  and visits each point of  $D(k)$  it has not visited yet, if any.

In particular,  $\tilde{\pi}$  visits all points  $\eta s(j)A$ ,  $0 \leq j \leq k$ . By Lemma 10 we get

$$\ell(\tilde{\pi}) \geq 8Ak.$$

As  $k \geq r/(20A)$ , we get

$$\ell(\tilde{\pi}) \geq r/4. \quad (17)$$

If a random ball  $B(c, r(c))$  disturbs a site  $s(j)$ , then

$$\|c - \eta s(j)A\| \leq 2A + r(c) \leq 3r(c). \quad (18)$$

We can easily give an upper bound on the length of  $\tilde{\pi}$  by considering the longer sequence in which in the definition "visits each point of  $D(j)$  it has not visited yet" is replaced by "goes back and forth between  $\eta s(j)A$  and points of  $D(j)$  it has not visited yet". Using (18) and Lemma 10, we get

$$\ell(\tilde{\pi}) \leq 12Ak + 2 \sum_{c \in D} 3r(c) = 12Ak + 6 \sum_{c \in D} r(c). \quad (19)$$

### 3.8 A lower bound on $T(r)$

Let us first check

$$T(r) = \tau(\pi) \geq \delta \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site and } D(j)=\emptyset}. \quad (20)$$

This is a consequence of the following facts.

- For any  $j \in \{0, \dots, k-1\}$ , the path  $\pi$  crosses the annulus  $B(\eta s(j)A, 2A) \setminus B(\eta s(j)A, A)$ . This is a consequence of Items 1 and 2 of Lemma 10 which yield, for any such  $j$ ,  $\pi(t(j)) \in B(\eta s(j)A, A)$  and  $\pi(t(j+1)) \notin B(\eta s(j)A, 2A)$ .
- The balls  $B(\eta s(j)A, 2A)$ ,  $j \in \{0, \dots, k-1\}$  are disjoint. Therefore the travel length of  $\pi$  is at least the sum of the travel length of  $\pi$  inside each annulus  $B(\eta s(j)A, 2A) \setminus B(\eta s(j)A, A)$ .
- For any  $j \in \{0, \dots, k-1\}$ , if  $D(j)$  is empty, then the travel time of  $\pi$  inside  $B(\eta s(j)A, 2A)$  does not change if we throw away random balls of radii at least  $A$ .
- If, in addition,  $s(j)$  is a good site, then the travel time of  $\pi$  inside  $B(\eta s(j)A, 2A)$  is at least  $\delta$ .

From (20) we deduce

$$T(r) \geq \delta \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} - \delta \sum_{j=0}^{k-1} 1_{D(j) \neq \emptyset}.$$

By Lemma 11, a given random ball  $B(c, r(c))$  such that  $r(c) \geq A$  disturbs at most  $2r(c)/A$  sites. Therefore

$$T(r) \geq \delta \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} - \delta \sum_{c \in D} \frac{2r(c)}{A}$$

where we recall that  $D$  is the union of  $D(j)$ ,  $j = 0 \dots k$ . Recall that any point of  $D$  is an element of the path  $\tilde{\pi}$  seen as a sequence. Note also that any point of  $D$  is the center of a ball of radius at least  $A$ . Therefore, using the notations of the greedy paths model,

$$\sum_{c \in D} r(c) \leq r_{\lambda \nu^A}(\tilde{\pi}) \quad (21)$$

where

$$\nu^A = \nu(\cdot \cap [A, +\infty[)$$

and where we use the natural coupling between the Poisson point process which defines the Boolean model and the Poisson point process which defines the above greedy paths model: we just throw away all point  $(c, r)$  such that  $r < A$ . Therefore

$$T(r) \geq \delta \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} - \frac{2\delta}{A} r_{\lambda \nu^A}(\tilde{\pi}).$$

At this point we would like to divide  $T(r)$  by  $r$ , the first sum on the right-hand side by  $k$  and the second sum by  $\ell(\tilde{\pi})$ . By definition of  $k$  (see (15)),  $r$  and  $k$  are closely related. The links between  $\ell(\tilde{\pi})$  and  $r$  (or  $k$ ) is less clear. Therefore we use the following trick.

$$T(r) \geq \delta \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} + \frac{24\delta}{A} r_{\lambda \nu^A}(\tilde{\pi}) - \frac{26\delta}{A} r_{\lambda \nu^A}(\tilde{\pi}).$$

By (17) we get

$$\frac{T(r)}{r} \geq \frac{T(r)}{4\ell(\tilde{\pi})}$$

and then

$$\begin{aligned} \frac{T(r)}{r} &\geq \delta \left( \frac{\sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} + \frac{24}{A} r_{\lambda\nu^A}(\tilde{\pi})}{4\ell(\tilde{\pi})} - \frac{\frac{26}{A} r_{\lambda\nu^A}(\tilde{\pi})}{4\ell(\tilde{\pi})} \right) \\ &= \frac{\delta}{A} \left( \frac{A \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} + 24r_{\lambda\nu^A}(\tilde{\pi})}{4\ell(\tilde{\pi})} - \frac{13}{2} \frac{r_{\lambda\nu^A}(\tilde{\pi})}{\ell(\tilde{\pi})} \right). \end{aligned}$$

Using (19) and (21) we then get

$$\frac{T(r)}{r} \geq \frac{\delta}{A} \left( \frac{A \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} + 24r_{\lambda\nu^A}(\tilde{\pi})}{48Ak + 24r_{\lambda\nu^A}(\tilde{\pi})} - \frac{13}{2} \frac{r_{\lambda\nu^A}(\tilde{\pi})}{\ell(\tilde{\pi})} \right).$$

Using

$$A \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}} \leq 48Ak$$

we obtain

$$\begin{aligned} \frac{T(r)}{r} &\geq \frac{\delta}{A} \left( \frac{A \sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}}}{48Ak} - \frac{13}{2} \frac{r_{\lambda\nu^A}(\tilde{\pi})}{\ell(\tilde{\pi})} \right) \\ &= \frac{\delta}{A} \left( \frac{\sum_{j=0}^{k-1} 1_{s(j) \text{ is a good site}}}{48k} - \frac{13}{2} \frac{r_{\lambda\nu^A}(\tilde{\pi})}{\ell(\tilde{\pi})} \right). \end{aligned}$$

Recall that  $\mathcal{S}(k)$  denotes the set of sequences  $(s'(0), \dots, s'(k-1))$  of distinct elements of  $\mathbb{Z}^d$  such that  $s'(0) = 0$  and, for any  $i \in \{1, \dots, k-1\}$ ,  $\|s'(i) - s'(i-1)\| \leq 12\eta^{-1}$ . Using also notations for greedy paths, we thus get

$$\frac{T(r)}{r} \geq \frac{\delta}{A} \left( \frac{\inf_{s' \in \mathcal{S}} \sum_{j=0}^{k-1} 1_{s'(j) \text{ is a good site}}}{48k} - \frac{13}{2} S_{\lambda\nu^A} \right).$$

By (9) we get

$$P \left( \frac{\inf_{s' \in \mathcal{S}} \sum_{j=0}^{k-1} 1_{s'(j) \text{ is a good site}}}{k} < \frac{1}{2} \right) \leq \frac{1}{2^k} \leq \frac{1}{2}.$$

The latter inequality follows from the definition of  $k$  (see (15)) and the assumption  $r \geq 20A$ . By Theorem 5 and by (10) we get

$$P(S_{\lambda\nu^A} \geq 1/4000) \leq 4000E(S_{\lambda\nu^A}) \leq 4000C \int_0^\infty \left( \lambda\nu_A([r, +\infty)) \right)^{1/d} dr \leq \frac{1}{4}.$$

Therefore

$$P\left(S_{\lambda\nu^A} \leq 1/4000 \text{ and } \frac{\inf_{s' \in \mathcal{S}} \sum_{j=0}^{k-1} 1_{s'(j) \text{ is a good site}}}{k} \geq \frac{1}{2}\right) \geq \frac{1}{4}$$

and therefore

$$P\left(\frac{T(r)}{r} \geq \frac{\delta}{200A}\right) \geq \frac{1}{4}.$$

As  $T(r)/r$  converges in probability to  $\mu$ , we get  $\mu > 0$ .

## Appendices

### A Openness of $\{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\}$ and positivity of $\widehat{\lambda}_c$

The aim of this section is to provide a proof of the following result. Recall that we assume (1).

**Theorem 12** *The set*

$$\{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\}$$

*is open and non-empty. In particular,  $\widehat{\lambda}_c$  is positive.*

The positivity of  $\widehat{\lambda}_c$  is implicit in [6]. The openness is a simple consequence of intermediate results in [6]. Both results are also consequences of Theorems 2.7 and 2.8 in [7] which deal with a more general framework.

We choose to give a proof using intermediate results in [6]. There is essentially no novelty in this section.

Let us recall some notation from [6]. Let  $\alpha > 0$ .

- $\Sigma(B(0, \alpha))$  is the union of random balls of the Boolean model with centers in  $B(0, \alpha)$ .
- $G(0, \alpha)$  is the event "there exists a path from  $S(\alpha)$  to  $S(8\alpha)$  in  $\Sigma(B(0, 10\alpha))$ ".
- $H(\alpha)$  is the event "there exists a random ball of the Boolean model with which touches  $B(0, 9\alpha)$  and whose center is outside  $B(0, 10\alpha)$ ".
- $\Pi(\alpha) = P(G(0, \alpha))$ .

The article [6] focus on the property  $\lim_{\alpha \rightarrow \infty} \Pi(\alpha) = 0$  while in this article we focus on the property  $\lim_{\alpha \rightarrow \infty} P(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) = 0$ . This is only a matter of taste, as shown by the first part of the following proposition. Set

$$\varepsilon(\alpha) = \int_{[\alpha, +\infty)} r^d \nu(dr). \quad (22)$$

Note that  $\lim_{\alpha \rightarrow \infty} \varepsilon(\alpha) = 0$ .

**Proposition 13** *There exists a constant  $K = K(d)$  such that, for any  $\alpha > 0$ ,*

$$\Pi(\alpha) \leq P(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) \leq K\Pi(\alpha/10) + \lambda K\varepsilon(\alpha/10), \quad (23)$$

$$\Pi(10\alpha) \leq K\Pi(\alpha)^2 + \lambda K\varepsilon(\alpha), \quad (24)$$

$$\Pi(\alpha) \leq \lambda K\alpha^d, \quad (25)$$

where  $\varepsilon$ , defined by (22), tends to 0 at  $\infty$ .

**Proof.** Inequalities (24) and (25) are part of Proposition 3.1 in [6]. Let us prove (23). Let  $A$  be a finite subset of  $S(1)$  such that  $S(1)$  is covered by the union of the balls  $B(a, 1/10)$ ,  $a \in A$ . Let  $K_1 = K_1(d)$  denote the cardinality of  $A$ .

Let  $\alpha > 0$ . The first inequality is straightforward as

$$G(0, \alpha) \subset \{S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)\}.$$

By definition of  $A$ ,  $S(\alpha)$  is covered by the union of the balls  $B(\alpha a, \alpha/10)$ ,  $a \in A$ . Moreover, all the balls  $B(\alpha a, 8\alpha/10)$ ,  $a \in A$ , are contained in  $B(0, 2\alpha)$ . Therefore

$$\{S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)\} \subset \bigcup_{a \in A} \{S(a, \alpha/10) \xleftrightarrow{\Sigma} S(a, 8\alpha/10)\}.$$

By union bound, by stationarity and by definition of  $K_1$  we get

$$P(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) \leq K_1 P(S(\alpha/10) \xleftrightarrow{\Sigma} S(8\alpha/10)).$$

Note

$$\{S(\alpha/10) \xleftrightarrow{\Sigma} S(8\alpha/10)\} \subset G(0, \alpha/10) \cup H(\alpha/10). \quad (26)$$

Indeed, assume the existence of a path in  $\Sigma$  from  $S(\alpha/10)$  to  $S(8\alpha/10)$ . We can assume that the path is in  $B(8\alpha/10)$ . If moreover  $H(\alpha/10)$  does not occur, then the path is in  $\Sigma(B(0, 10\alpha/10))$  and thus  $G(0, \alpha/10)$  occurs. Therefore,

$$P(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) \leq K_1 \Pi(\alpha/10) + K_1 P(H(\alpha/10)).$$

But there exists a constant  $K_2 = K_2(d)$  such that,

$$P(H(\alpha/10)) \leq \lambda K_2 \varepsilon(\alpha/10).$$

This is Lemma 3.4 in [6]. The lemma follows.  $\square$

Inequality (24) yields the following result.

**Lemma 14** *We use the constant  $K$  from the previous lemma. Let  $M > 0$ . Assume*

$$\lambda K^2 \varepsilon(M) \leq \frac{1}{4} \quad (27)$$

and, for all  $\alpha \in [M, 10M]$ ,

$$K\Pi(\alpha) \leq \frac{1}{2}. \quad (28)$$

Then  $\lim_{\alpha \rightarrow \infty} \Pi(\alpha) = 0$ .

**Proof.** This is a consequence of (24) and Lemma 3.7 in [6]. Showing how to apply Lemma 3.7 would not be much shorter than adapting the proof in our context. Therefore we choose to give a full proof. By (24), for all  $\alpha > 0$ ,

$$K\Pi(10\alpha) \leq (K\Pi(\alpha))^2 + \lambda K^2 \varepsilon(\alpha). \quad (29)$$

As  $\varepsilon$  is non-increasing, (27) yields, for all  $\alpha \geq M$ ,

$$K\Pi(10\alpha) \leq (K\Pi(\alpha))^2 + \frac{1}{4}.$$

Therefore, if moreover  $K\Pi(\alpha) \leq 1/2$ , then  $K\Pi(10\alpha) \leq 1/2$ . Using (28) and induction we deduce, for all  $\alpha \geq M$ ,  $K\Pi(\alpha) \leq 1/2$ . As a consequence,

$$\limsup_{\alpha \rightarrow \infty} K\Pi(\alpha) \leq \frac{1}{2}.$$

As  $\varepsilon$  tends to 0 at  $\infty$  we get, using (29),

$$\limsup_{\alpha \rightarrow \infty} K\Pi(\alpha) \leq \left( \limsup_{\alpha \rightarrow \infty} K\Pi(\alpha) \right)^2.$$

As a consequence of the two previous inequalities we get  $\limsup_{\alpha \rightarrow \infty} K\Pi(\alpha) = 0$ .  $\square$

**Proof of Theorem 12.** By (23),

$$I = \{\lambda > 0 : \lim_{r \rightarrow \infty} P(S(r) \xleftrightarrow{\Sigma} S(2r)) = 0\} = \{\lambda > 0 : \lim_{\alpha \rightarrow \infty} \Pi(\alpha) = 0\}.$$

Let us first prove that  $I$  is non empty. Set  $M = 1$ . By (25), for small enough  $\lambda > 0$ , Assumptions (27) and (28) hold for every  $\alpha \in [1, 10]$ . By Lemma 14, all such  $\alpha$  belong to  $I$ .

Let us now prove that  $I$  is open. If  $\lambda$  belongs to  $I$ , then any smaller positive real number belongs to  $I$ . Therefore, we only have to show that, for any  $\lambda \in I$ , there exists  $\eta > 0$  such that  $\lambda + \eta \in I$ . We now fix  $\lambda \in I$ . Note that  $\varepsilon$  and  $K$  does not depend on the density  $\lambda$ . We emphasize the dependence of  $\Pi$  on  $\lambda$  by writing  $\Pi_\lambda$ . As  $\Pi_\lambda$  and  $\varepsilon$  tends to 0 at infinity we can fix  $M > 0$  such that

$$(\lambda + 1)K^2\varepsilon(M) \leq \frac{1}{4} \quad (30)$$

and, for all  $\alpha \in [M, 10M]$ ,

$$K\Pi_\lambda(\alpha) \leq \frac{1}{4}.$$

Let  $\eta > 0$ . Consider a Boolean model  $\Sigma'$  with parameters  $(\eta, \nu, d)$  independent of  $\Sigma$ . Then  $\Sigma \cup \Sigma'$  is a Boolean model with parameters  $(\lambda + \eta, \nu, d)$ . Therefore, for any  $\alpha \in [M, 10M]$ ,

$$\begin{aligned} \Pi_{\lambda+\eta}(\alpha) &\leq \Pi_\lambda(\alpha) + P(\text{one of the random balls of } \Sigma' \text{ is centered in } B(0, 10\alpha)) \\ &\leq \Pi_\lambda(\alpha) + \eta v_d (10\alpha)^d \\ &\leq \Pi_\lambda(\alpha) + \eta v_d (100M)^d \end{aligned}$$

where  $v_d$  denotes the volume of the unit ball of  $\mathbb{R}^d$ . As a consequence, for  $\eta \in (0, 1)$  small enough, for any  $\alpha \in [M, 10M]$ ,

$$K\Pi_{\lambda+\eta}(\alpha) \leq \frac{1}{2}.$$

From (30) and  $\eta \leq 1$  we also get

$$(\lambda + \eta)K^2\varepsilon(M) \leq \frac{1}{4}.$$

By Lemma 14, we then deduce the convergence of  $\Pi_{\lambda+\eta}(\alpha)$  to 0 as  $\alpha$  tends to  $\infty$ . In other words,  $\lambda + \eta$  belongs to  $I$ .  $\square$

## B Asymptotic behaviour of $T(x)/\|x\|$

In this section we prove Theorem 1. The plan of proof is standard and the proof is actually particularly simple thanks to good upper bounds on  $T$ . Let us first state Kingman's theorem. We choose to state this theorem as Kesten in [10] (see Theorem 2.1), following the statement proposed by Liggett in [11].

**Theorem 15** *Suppose  $(X_{m,n}, 0 \leq m < n)$  ( $m$  and  $n$  are integer) is a family of random variables satisfying:*

1. *For all integers  $m, n$  such that  $0 < m < n$ , one has  $X_{0,n} \leq X_{0,m} + X_{m,n}$ ,*
2. *For each  $m \geq 0$ , the distribution of  $(X_{m+h,m+h+k}, k \geq 1)$  does not depend on the integer  $h \geq 0$ ,*
3. *For each  $k \geq 1$ , the sequence  $(X_{nk,(n+1)k}, n \geq 0)$  is stationary and ergodic,*
4.  *$E(X_{0,1}^+) < \infty$  and there exists a real  $c$  such that, for all natural integer  $n$ , one has  $E(X_{0,n}) \geq -cn$ .*

Then

$$\lim_{n \rightarrow \infty} \frac{X_{0,n}}{n} = \gamma \text{ a.s. and in } L^1$$

where  $\gamma$  is the finite constant defined by

$$\gamma = \inf_n \frac{E(X_{0,n})}{n}.$$

Let  $x \in S(1)$ . We apply Kingman's theorem to the family defined by  $X_{m,n} = T(mx, nx)$ .

- For any  $a, b, c \in \mathbb{R}^d$ ,  $T(a, c) \leq T(a, b) + T(b, c)$ . This follows from the fact that the concatenation of a path from  $a$  to  $b$  and a path from  $b$  to  $c$  is a path from  $a$  to  $c$ . Therefore the first assumption of Kingman's theorem holds.
- The process  $X$  is stationary and ergodic under the action of spatial translations. Therefore the second and third assumptions of Kingman's theorem hold.
- For any  $a, b \in \mathbb{R}^d$ ,

$$0 \leq T(a, b) \leq \|b - a\|. \tag{31}$$

Therefore the forth assumption holds.

Thus,

$$\lim_{n \rightarrow \infty} \frac{T(0, nx)}{n} = \mu(x) \text{ a.s. and in } L^1.$$

By isotropy of the model, we get that  $\mu$  does not depend on  $x \in S(1)$ . Therefore we drop the dependence on  $x$  and write  $\mu$ . We have proven

$$\text{for all } x \in S(1), \lim_{n \rightarrow \infty} \frac{T(0, nx)}{n} = \mu(x) \text{ a.s. and in } L^1. \quad (32)$$

Now we prove the uniformity of the convergence. For any real  $u$ , we denote by  $\lfloor u \rfloor$  its integer part and by  $\{u\}$  its fractional part. In particular,  $u = \lfloor u \rfloor + \{u\}$ . Let  $\varepsilon > 0$ . Let  $A$  be a finite subset of  $S(1)$  such that any point of  $S(1)$  is at most at distance  $\varepsilon$  of some point of  $A$ . By (32), with probability one, there exists  $N$  such that for any  $n \geq N$  and for any  $x \in A$ ,

$$\left| \frac{T(0, nx)}{n} - \mu \right| \leq \varepsilon.$$

Let  $y \in \mathbb{R}^d \setminus \{0\}$ . Write

$$\hat{y} = \frac{1}{\|y\|} y \text{ and } n(y) = \lfloor \|y\| \rfloor. \quad (33)$$

We assume  $n(y) \geq N$  and  $n(y)\varepsilon \geq 1$ . Let  $x \in A$  be such that  $\|\hat{y} - x\| \leq \varepsilon$ . By triangle inequality for  $T$  and by (31) we get

$$\begin{aligned} |T(0, y) - T(0, n(y)x)| &\leq |T(0, y) - T(0, n(y)\hat{y})| + |T(0, n(y)\hat{y}) - T(0, n(y)x)| \\ &\leq \|y - n(y)\hat{y}\| + \|n(y)\hat{y} - n(y)x\| \\ &\leq 1 + n(y)\varepsilon \\ &\leq 2n(y)\varepsilon. \end{aligned}$$

Moreover, as  $n(y) \geq N$ , we also have  $|T(0, n(y)x) - n(y)\mu| \leq n(y)\varepsilon$ . Therefore

$$\frac{|T(0, y) - n(y)\mu|}{n(y)} \leq 3\varepsilon.$$

The almost sure convergence in Theorem 1 follows. The convergence in  $L^1$  is a straightforward consequence of the a.s. convergence and the dominated convergence with the domination  $T(0, y)/\|y\| \leq 1$  for every  $y \in \mathbb{R}^d \setminus \{0\}$ .

## References

- [1] Daniel Ahlberg, Vincent Tassion, and Augusto Teixeira. Sharpness of the phase transition for continuum percolation in  $r^2$ . Available on Arxiv, 2016.
- [2] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.
- [3] Maria Deijfen. Asymptotic shape in a continuum growth model. *Adv. in Appl. Probab.*, 35(2):303–318, 2003.



- [4] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for bernoulli percolation on  $z^d$ . To appear in *Enseign. Math.*, 2015.
- [5] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for bernoulli percolation and the ising model. *Communications in Mathematical Physics*, 343(2):725–745, 2016.
- [6] Jean-Baptiste Gou    . Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab.*, 36(4):1209–1220, 2008.
- [7] Jean-Baptiste Gou    . Subcritical regimes in some models of continuum percolation. *Ann. Appl. Probab.*, 19(4):1292–1318, 2009.
- [8] Jean-Baptiste Gou     and R       Marchand. Continuous first-passage percolation and continuous greedy paths model: linear growth. *Ann. Appl. Probab.*, 18(6):2300–2319, 2008.
- [9] Peter Hall. On continuum percolation. *Ann. Probab.*, 13(4):1250–1266, 1985.
- [10] Harry Kesten. Aspects of first passage percolation. In *       d       de probabilit     de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 125–264. Springer, Berlin, 1986.
- [11] Thomas M. Liggett. An improved subadditive ergodic theorem. *Ann. Probab.*, 13(4):1279–1285, 1985.
- [12] James B. Martin. Linear growth for greedy lattice animals. *Stochastic Process. Appl.*, 98(1):43–66, 2002.
- [13] Ronald Meester and Rahul Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [14] M. V. Men’shikov. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, 288(6):1308–1311, 1986.
- [15] S. Ziesche. Sharpness of the phase transition and lower bounds for the critical intensity in continuum percolation on  $\mathbb{R}^d$ . *ArXiv e-prints*, July 2016.